AQA Maths Further Pure 3

Past Paper Pack

2006-2015

General Certificate of Education
January 2006
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

ASSESSMENT and QUALIFICATIONS ALLIANCE

Friday 27 January 2006 1.30 pm to 3.00 pm

For this paper you must have:

- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions

- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.

Information

- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice

- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 (a) Show that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$$ *(2 marks)*

(b) Hence find the sum of the first n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots$$ *(4 marks)*

2 The cubic equation

$$x^3 + px^2 + qx + r = 0$$

where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

find the values of p and q. *(5 marks)*

(b) Given further that one root is $3 + \mathrm{i}$, find the value of r. *(5 marks)*

3 The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \qquad \text{and} \qquad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

(a) Show that $z_1 = \mathrm{i}$. *(2 marks)*

(b) Show that $|z_1| = |z_2|$. *(2 marks)*

(c) Express both z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(d) Draw an Argand diagram to show the points representing z_1, z_2 and $z_1 + z_2$. *(2 marks)*

(e) Use your Argand diagram to show that

$$\tan \frac{5}{12}\pi = 2 + \sqrt{3}$$ *(3 marks)*

4 (a) Prove by induction that

$$2 + (3 \times 2) + (4 \times 2^2) + \ldots + (n+1)\,2^{n-1} = n\,2^n$$

for all integers $n \geqslant 1$. *(6 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} (r+1)\,2^{r-1} = n\,2^n(2^{n+1} - 1)$$ *(3 marks)*

5 The complex number z satisfies the relation

$$|z + 4 - 4\mathrm{i}| = 4$$

(a) Sketch, on an Argand diagram, the locus of z. *(3 marks)*

(b) Show that the greatest value of $|z|$ is $4(\sqrt{2} + 1)$. *(3 marks)*

(c) Find the value of z for which

$$\arg(z + 4 - 4\mathrm{i}) = \tfrac{1}{6}\pi$$

Give your answer in the form $a + \mathrm{i}b$. *(3 marks)*

Turn over for the next question

6 It is given that $z = \mathrm{e}^{\mathrm{i}\theta}$.

(a) (i) Show that

$$z + \frac{1}{z} = 2\cos\theta$$ *(2 marks)*

(ii) Find a similar expression for

$$z^2 + \frac{1}{z^2}$$ *(2 marks)*

(iii) Hence show that

$$z^2 - z + 2 - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta$$ *(3 marks)*

(b) Hence solve the quartic equation

$$z^4 - z^3 + 2z^2 - z + 1 = 0$$

giving the roots in the form $a + \mathrm{i}b$. *(5 marks)*

7 (a) Use the definitions

$$\sinh\theta = \frac{1}{2}(\mathrm{e}^{\theta} - \mathrm{e}^{-\theta}) \qquad \text{and} \qquad \cosh\theta = \frac{1}{2}(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta = 0$ to the point where $\theta = 1$ is

$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}} - 1\right]$$ *(6 marks)*

END OF QUESTIONS

General Certificate of Education
June 2006
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Monday 19 June 2006 9.00 am to 10.30 am

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 It is given that y satisfies the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x - 10 - 10\cos 2x$$

(a) Show that $y = 2x + \sin 2x$ is a particular integral of the given differential equation. *(3 marks)*

(b) Find the general solution of the differential equation. *(4 marks)*

(c) Hence express y in terms of x, given that $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$. *(4 marks)*

2 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \frac{x^2 + y^2}{xy}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(6 marks)*

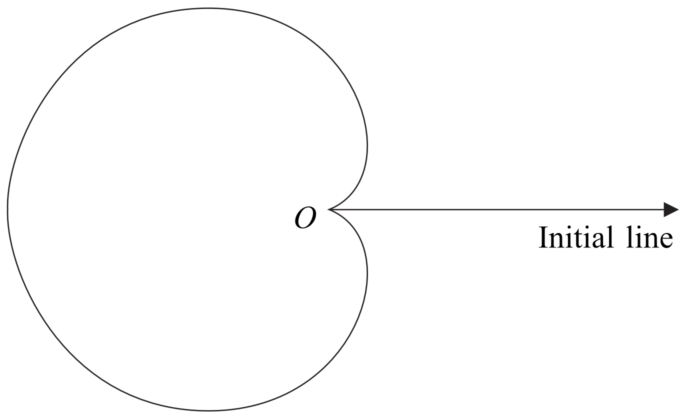
3 (a) Show that $\sin x$ is an integrating factor for the differential equation

$$\frac{dy}{dx} + (\cot x)y = 2\cos x$$ *(3 marks)*

(b) Solve this differential equation, given that $y = 2$ when $x = \frac{\pi}{2}$. *(6 marks)*

4 The diagram shows the curve C with polar equation

$$r = 6(1 - \cos\theta), \qquad 0 \leqslant \theta < 2\pi$$



(a) Find the area of the region bounded by the curve C. *(6 marks)*

(b) The circle with cartesian equation $x^2 + y^2 = 9$ intersects the curve C at the points A and B.

(i) Find the polar coordinates of A and B. *(4 marks)*

(ii) Find, in surd form, the length of AB. *(2 marks)*

5 (a) Show that $\lim_{a\to\infty}\left(\frac{3a+2}{2a+3}\right) = \frac{3}{2}$. *(2 marks)*

(b) Evaluate $\int_1^\infty \left(\frac{3}{3x+2} - \frac{2}{2x+3}\right) dx$, giving your answer in the form $\ln k$, where k is a rational number. *(5 marks)*

6 (a) Show that the substitution

$$u = \frac{\mathrm{d}y}{\mathrm{d}x} + 2y$$

transforms the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \mathrm{e}^{-2x}$$

into

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2u = \mathrm{e}^{-2x}$$

(4 marks)

(b) By using an integrating factor, or otherwise, find the general solution of

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2u = \mathrm{e}^{-2x}$$

giving your answer in the form $u = \mathrm{f}(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \mathrm{e}^{-2x}$$

giving your answer in the form $y = \mathrm{g}(x)$. *(5 marks)*

7 (a) (i) Write down the first three terms of the binomial expansion of $(1+y)^{-1}$, in ascending powers of y. *(1 mark)*

(ii) By using the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

and your answer to part (a)(i), or otherwise, show that the first three non-zero terms in the expansion, in ascending powers of x, of $\sec x$ are

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}$$ *(5 marks)*

(b) By using Maclaurin's theorem, or otherwise, show that the first two non-zero terms in the expansion, in ascending powers of x, of $\tan x$ are

$$x + \frac{x^3}{3}$$ *(3 marks)*

(c) Hence find $\lim_{x \to 0} \left(\frac{x \tan 2x}{\sec x - 1} \right)$. *(4 marks)*

END OF QUESTIONS

General Certificate of Education
January 2007
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Friday 26 January 2007 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \ln(1 + x^2 + y)$$

and $$y(1) = 0.6$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$$

with $h = 0.05$, to obtain an approximation to $y(1.05)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\,\mathrm{f}(x_r, y_r)$ and $k_2 = h\,\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.05$, to obtain an approximation to $y(1.05)$, giving your answer to four decimal places. *(6 marks)*

2 A curve has polar equation $r(1 - \sin\theta) = 4$. Find its cartesian equation in the form $y = \mathrm{f}(x)$. *(6 marks)*

3 (a) Show that x^2 is an integrating factor for the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 3(x^3 + 1)^{\frac{1}{2}}$$ *(3 marks)*

(b) Solve this differential equation, given that $y = 1$ when $x = 2$. *(6 marks)*

4 (a) Explain why $\int_0^{e} \frac{\ln x}{\sqrt{x}}\, dx$ is an improper integral. *(1 mark)*

(b) Use integration by parts to find $\int x^{-\frac{1}{2}} \ln x \, dx$. *(3 marks)*

(c) Show that $\int_0^{e} \frac{\ln x}{\sqrt{x}}\, dx$ exists and find its value. *(4 marks)*

5 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6 + 5\sin x$$ *(12 marks)*

6 The function f is defined by $f(x) = (1 + 2x)^{\frac{1}{2}}$.

(a) (i) Find $f'''(x)$. *(4 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of x,

$$f(x) \approx 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$ *(4 marks)*

(b) Use the expansion of e^x together with the result in part (a)(ii) to show that, for small values of x,

$$e^x(1 + 2x)^{\frac{1}{2}} \approx 1 + 2x + x^2 + kx^3$$

where k is a rational number to be found. *(3 marks)*

(c) Write down the first four terms in the expansion, in ascending powers of x, of e^{2x}. *(1 mark)*

(d) Find

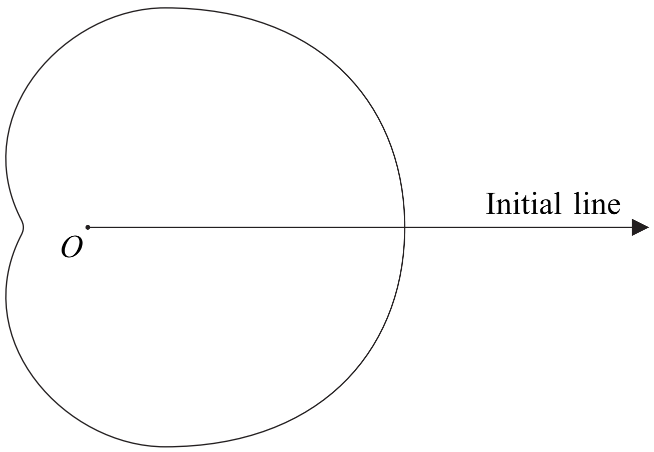
$$\lim_{x \to 0} \frac{e^x(1 + 2x)^{\frac{1}{2}} - e^{2x}}{1 - \cos x}$$ *(4 marks)*

Turn over for the next question

7 A curve C has polar equation

$$r = 6 + 4\cos\theta, \qquad -\pi \leqslant \theta \leqslant \pi$$

The diagram shows a sketch of the curve C, the pole O and the initial line.



(a) Calculate the area of the region bounded by the curve C. *(6 marks)*

(b) The point P is the point on the curve C for which $\theta = \frac{2\pi}{3}$.

The point Q is the point on C for which $\theta = \pi$.

Show that QP is parallel to the line $\theta = \frac{\pi}{2}$. *(4 marks)*

(c) The line PQ intersects the curve C again at a point R.

The line RO intersects C again at a point S.

(i) Find, in surd form, the length of PS. *(4 marks)*

(ii) Show that the angle OPS is a right angle. *(1 mark)*

END OF QUESTIONS

General Certificate of Education
June 2007
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Wednesday 20 June 2007 1.30 pm to 3.00 pm

For this paper you must have: • an 8-page answer book • the **blue** AQA booklet of formulae and statistical tables. You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions

- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information

- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice

- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Find the value of the constant k for which kx^2e^{5x} is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$$ *(6 marks)*

(b) Hence find the general solution of this differential equation. *(4 marks)*

2 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \sqrt{x^2 + y^2 + 3}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(6 marks)*

3 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

given that $y = 3$ when $x = 0$. *(8 marks)*

4 (a) Show that $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$. *(1 mark)*

(b) A curve has cartesian equation

$$(x^2 + y^2)^3 = (x + y)^4$$

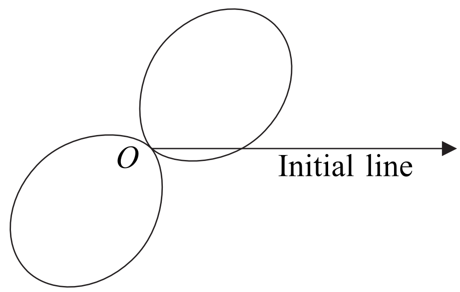
Given that $r \geqslant 0$, show that the polar equation of the curve is

$$r = 1 + \sin 2\theta$$ *(4 marks)*

(c) The curve with polar equation

$$r = 1 + \sin 2\theta, \quad -\pi \leqslant \theta \leqslant \pi$$

is shown in the diagram.



(i) Find the two values of θ for which $r = 0$. *(3 marks)*

(ii) Find the area of one of the loops. *(6 marks)*

Turn over for the next question

5 (a) A differential equation is given by

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

Show that the substitution

$$u = \frac{dy}{dx} + x$$

transforms this differential equation into

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

(4 marks)

(b) Find the general solution of

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

giving your answer in the form $y = g(x)$. *(3 marks)*

6 (a) The function f is defined by

$$f(x) = \ln(1 + e^x)$$

Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$\ln 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(b) Hence write down the first two non-zero terms in the expansion, in ascending powers of x, of $\ln\left(\dfrac{1 + e^x}{2}\right)$. *(1 mark)*

(c) Use the series expansion

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

to write down the first three terms in the expansion, in ascending powers of x, of $\ln\left(1 - \dfrac{x}{2}\right)$. *(1 mark)*

(d) Use your answers to parts (b) and (c) to find

$$\lim_{x \to 0} \left[\frac{\ln\left(\dfrac{1 + e^x}{2}\right) + \ln\left(1 - \dfrac{x}{2}\right)}{x - \sin x} \right]$$ *(4 marks)*

7 (a) Write down the value of

$$\lim_{x \to \infty} x e^{-x}$$ *(1 mark)*

(b) Use the substitution $u = xe^{-x} + 1$ to find $\displaystyle\int \frac{e^{-x}(1 - x)}{xe^{-x} + 1}\, dx$. *(2 marks)*

(c) Hence evaluate $\displaystyle\int_1^{\infty} \frac{1 - x}{x + e^x}\, dx$, showing the limiting process used. *(4 marks)*

END OF QUESTIONS

General Certificate of Education
January 2008
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Friday 25 January 2008 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = x^2 - y^2$$

and $$y(2) = 1$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

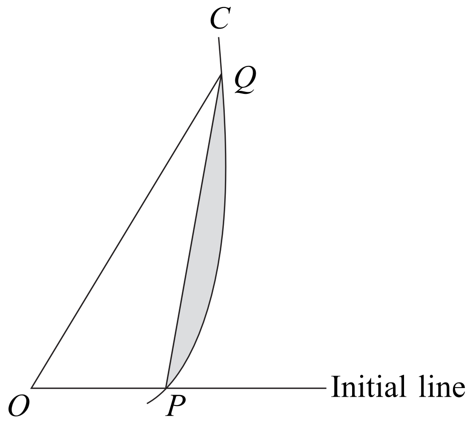
with $h = 0.1$, to obtain an approximation to $y(2.1)$. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\mathrm{f}(x_r, y_r)$$

with your answer to part (a), to obtain an approximation to $y(2.2)$. *(3 marks)*

2 The diagram shows a sketch of part of the curve C whose polar equation is $r = 1 + \tan\theta$. The point O is the pole.



The points P and Q on the curve are given by $\theta = 0$ and $\theta = \frac{\pi}{3}$ respectively.

(a) Show that the area of the region bounded by the curve C and the lines OP and OQ is

$$\frac{1}{2}\sqrt{3} + \ln 2$$ *(6 marks)*

(b) Hence find the area of the shaded region bounded by the line PQ and the arc PQ of C. *(3 marks)*

3 (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5$$ *(6 marks)*

(b) Hence express y in terms of x, given that $y = 2$ and $\frac{dy}{dx} = 3$ when $x = 0$. *(4 marks)*

4 (a) Explain why $\int_1^\infty xe^{-3x}\,dx$ is an improper integral. *(1 mark)*

(b) Find $\int xe^{-3x}\,dx$. *(3 marks)*

(c) Hence evaluate $\int_1^\infty xe^{-3x}\,dx$, showing the limiting process used. *(3 marks)*

Turn over ▶

5 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + \frac{4x}{x^2+1}y = x$$

given that $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. *(9 marks)*

6 A curve C has polar equation

$$r^2 \sin 2\theta = 8$$

(a) Find the cartesian equation of C in the form $y = f(x)$. *(3 marks)*

(b) Sketch the curve C. *(1 mark)*

(c) The line with polar equation $r = 2\sec\theta$ intersects C at the point A. Find the polar coordinates of A. *(4 marks)*

7 (a) (i) Write down the expansion of $\ln(1+2x)$ in ascending powers of x up to and including the term in x^3. *(2 marks)*

(ii) State the range of values of x for which this expansion is valid. *(1 mark)*

(b) (i) Given that $y = \ln\cos x$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. *(4 marks)*

(ii) Find the value of $\frac{d^4y}{dx^4}$ when $x = 0$. *(3 marks)*

(iii) Hence, by using Maclaurin's theorem, show that the first two non-zero terms in the expansion, in ascending powers of x, of $\ln\cos x$ are

$$-\frac{x^2}{2} - \frac{x^4}{12}$$ *(2 marks)*

(c) Find

$$\lim_{x\to 0}\left[\frac{x\ln(1+2x)}{x^2 - \ln\cos x}\right]$$ *(3 marks)*

8 (a) Given that $x = \mathrm{e}^t$ and that y is a function of x, show that:

(i) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t}$; *(3 marks)*

(ii) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - \dfrac{\mathrm{d}y}{\mathrm{d}t}$. *(3 marks)*

(b) Hence find the general solution of the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

(5 marks)

END OF QUESTIONS

General Certificate of Education
June 2008
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Monday 16 June 2008 1.30 pm to 3.00 pm

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \ln(x + y)$$

and $$y(2) = 3$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(2.1)$, giving your answer to four decimal places. *(6 marks)*

2 (a) Find the values of the constants a, b, c and d for which $a + bx + c\sin x + d\cos x$ is a particular integral of the differential equation

$$\frac{dy}{dx} - 3y = 10\sin x - 3x$$ *(4 marks)*

(b) Hence find the general solution of this differential equation. *(3 marks)*

3 (a) Show that $x^2 = 1 - 2y$ can be written in the form $x^2 + y^2 = (1 - y)^2$. *(1 mark)*

(b) A curve has cartesian equation $x^2 = 1 - 2y$.

Find its polar equation in the form $r = f(\theta)$, given that $r > 0$. *(5 marks)*

4 (a) A differential equation is given by

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

Show that the substitution

$$u = \frac{dy}{dx}$$

transforms this differential equation into

$$\frac{du}{dx} - \frac{1}{x}u = 3x$$ *(2 marks)*

(b) By using an integrating factor, find the general solution of

$$\frac{du}{dx} - \frac{1}{x}u = 3x$$

giving your answer in the form $u = f(x)$. *(6 marks)*

(c) Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2$$

giving your answer in the form $y = g(x)$. *(2 marks)*

5 (a) Find $\int x^3 \ln x \, dx$. *(3 marks)*

(b) Explain why $\int_0^e x^3 \ln x \, dx$ is an improper integral. *(1 mark)*

(c) Evaluate $\int_0^e x^3 \ln x \, dx$, showing the limiting process used. *(3 marks)*

6 (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 10e^{-2x} - 9$$ *(10 marks)*

(b) Hence express y in terms of x, given that $y = 7$ when $x = 0$ and that $\frac{dy}{dx} \to 0$ as $x \to \infty$. *(4 marks)*

Turn over for the next question

7 (a) Write down the expansion of $\sin 2x$ in ascending powers of x up to and including the term in x^3. *(1 mark)*

(b) (i) Given that $y = \sqrt{3 + e^x}$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 0$. *(5 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of x,

$$\sqrt{3 + e^x} \approx 2 + \tfrac{1}{4}x + \tfrac{7}{64}x^2$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[\frac{\sqrt{3 + e^x} - 2}{\sin 2x} \right]$$ *(3 marks)*

8 The polar equation of a curve C is

$$r = 5 + 2\cos\theta, \qquad -\pi \leqslant \theta \leqslant \pi$$

(a) Verify that the points A and B, with **polar coordinates** $(7, 0)$ and $(3, \pi)$ respectively, lie on the curve C. *(2 marks)*

(b) Sketch the curve C. *(2 marks)*

(c) Find the area of the region bounded by the curve C. *(6 marks)*

(d) The point P is the point on the curve C for which $\theta = \alpha$, where $0 < \alpha \leqslant \frac{\pi}{2}$. The point Q lies on the curve such that POQ is a straight line, where the point O is the pole. Find, in terms of α, the area of triangle OQB. *(4 marks)*

END OF QUESTIONS

General Certificate of Education
January 2009
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Wednesday 21 January 2009 1.30 pm to 3.00 pm

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \frac{x^2 + y^2}{x + y}$$

and $$y(1) = 3$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

with $h = 0.2$, to obtain an approximation to $y(1.2)$. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

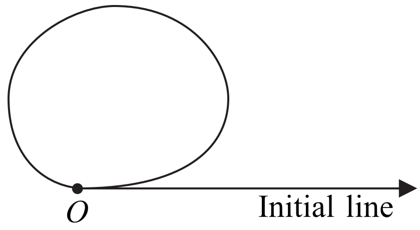
where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.2$, to obtain an approximation to $y(1.2)$, giving your answer to four decimal places. *(5 marks)*

2 (a) Show that $\frac{1}{x^2}$ is an integrating factor for the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = x$$ *(3 marks)*

(b) Hence find the general solution of this differential equation, giving your answer in the form $y = \mathrm{f}(x)$. *(4 marks)*

3 The diagram shows a sketch of a loop, the pole O and the initial line.



The polar equation of the loop is

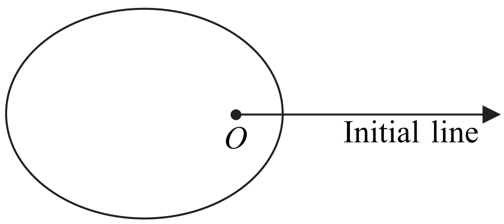
$$r = (2 + \cos\theta)\sqrt{\sin\theta}, \quad 0 \leqslant \theta \leqslant \pi$$

Find the area enclosed by the loop. *(6 marks)*

4 (a) Use integration by parts to show that $\int \ln x \, dx = x \ln x - x + c$, where c is an arbitrary constant. *(2 marks)*

(b) Hence evaluate $\int_0^1 \ln x \, dx$, showing the limiting process used. *(4 marks)*

5 The diagram shows a sketch of a curve C, the pole O and the initial line.



The curve C has polar equation

$$r = \frac{2}{3 + 2\cos\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$$

(a) Verify that the point L with polar coordinates $(2, \pi)$ lies on C. *(1 mark)*

(b) The circle with polar equation $r = 1$ intersects C at the points M and N.

(i) Find the polar coordinates of M and N. *(3 marks)*

(ii) Find the area of triangle LMN. *(4 marks)*

(c) Find a cartesian equation of C, giving your answer in the form $9y^2 = f(x)$. *(5 marks)*

Turn over for the next question

Turn over ▶

6 The function f is defined by $\mathrm{f}(x) = \mathrm{e}^{2x}(1+3x)^{-\frac{2}{3}}$.

(a) (i) Use the series expansion for e^x to write down the first four terms in the series expansion of e^{2x}. *(2 marks)*

(ii) Use the binomial series expansion of $(1+3x)^{-\frac{2}{3}}$ and your answer to part (a)(i) to show that the first three non-zero terms in the series expansion of $\mathrm{f}(x)$ are $1 + 3x^2 - 6x^3$. *(5 marks)*

(b) (i) Given that $y = \ln(1 + 2\sin x)$, find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of x,

$$\ln(1 + 2\sin x) \approx 2x - 2x^2$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \frac{1 - \mathrm{f}(x)}{x \ln(1 + 2\sin x)}$$ *(3 marks)*

7 (a) Given that $x = \mathrm{e}^t$ and that y is a function of x, show that

$$x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}$$ *(7 marks)*

(b) Hence show that the substitution $x = \mathrm{e}^t$ transforms the differential equation

$$x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 10$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} = 10$$ *(2 marks)*

(c) Find the general solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\dfrac{\mathrm{d}y}{\mathrm{d}t} = 10$. *(5 marks)*

(d) Hence solve the differential equation $x^2 \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 10$, given that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 8$ when $x = 1$. *(5 marks)*

END OF QUESTIONS

General Certificate of Education
June 2009
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Thursday 11 June 2009 9.00 am to 10.30 am

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \sqrt{x^2 + y + 1}$$

and $$y(3) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$$

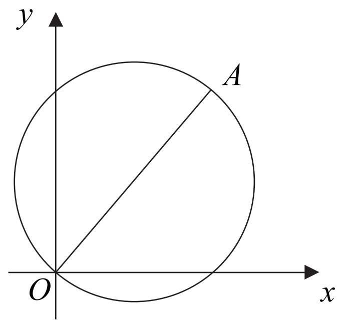
with your answer to part (a), to obtain an approximation to $y(3.2)$, giving your answer to three decimal places. *(3 marks)*

2 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} - y\tan x = 2\sin x$$

given that $y = 2$ when $x = 0$. *(9 marks)*

3 The diagram shows a sketch of a circle which passes through the origin O.



The equation of the circle is $(x-3)^2+(y-4)^2=25$ and OA is a diameter.

(a) Find the cartesian coordinates of the point A. *(2 marks)*

(b) Using O as the pole and the positive x-axis as the initial line, the polar coordinates of A are (k, α).

(i) Find the value of k and the value of $\tan\alpha$. *(2 marks)*

(ii) Find the polar equation of the circle $(x-3)^2+(y-4)^2=25$, giving your answer in the form $r=p\cos\theta+q\sin\theta$. *(4 marks)*

4 Evaluate the improper integral

$$\int_1^{\infty}\left(\frac{1}{x}-\frac{4}{4x+1}\right)\mathrm{d}x$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant to be found. *(5 marks)*

5 It is given that y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+5y=8\sin x+4\cos x$$

(a) Find the value of the constant k for which $y=k\sin x$ is a particular integral of the given differential equation. *(3 marks)*

(b) Solve the differential equation, expressing y in terms of x, given that $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=4$ when $x=0$. *(8 marks)*

Turn over ▶

6 The function f is defined by

$$f(x) = (9 + \tan x)^{\frac{1}{2}}$$

(a) (i) Find $f''(x)$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of x,

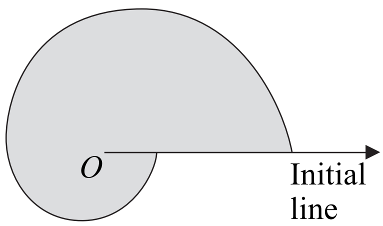
$$(9 + \tan x)^{\frac{1}{2}} \approx 3 + \frac{x}{6} - \frac{x^2}{216}$$ *(3 marks)*

(b) Find

$$\lim_{x \to 0} \left[\frac{f(x) - 3}{\sin 3x}\right]$$ *(3 marks)*

7 The diagram shows the curve C_1 with polar equation

$$r = 1 + 6e^{-\frac{\theta}{\pi}}, \quad 0 \leqslant \theta \leqslant 2\pi$$



(a) Find, in terms of π and e, the area of the shaded region bounded by C_1 and the initial line. *(5 marks)*

(b) The polar equation of a curve C_2 is

$$r = e^{\frac{\theta}{\pi}}, \quad 0 \leqslant \theta \leqslant 2\pi$$

Sketch the curve C_2 and state the polar coordinates of the end-points of this curve. *(4 marks)*

(c) The curves C_1 and C_2 intersect at the point P. Find the polar coordinates of P. *(5 marks)*

8 (a) Given that $x = t^2$, where $t \geqslant 0$, and that y is a function of x, show that:

(i) $2\sqrt{x}\,\dfrac{dy}{dx} = \dfrac{dy}{dt}$; *(3 marks)*

(ii) $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = \dfrac{d^2y}{dt^2}$. *(3 marks)*

(b) Hence show that the substitution $x = t^2$, where $t \geqslant 0$, transforms the differential equation

$$4x\frac{d^2y}{dx^2} + 2(1 + 2\sqrt{x})\frac{dy}{dx} - 3y = 0$$

into

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$$ *(2 marks)*

(c) Hence find the general solution of the differential equation

$$4x\frac{d^2y}{dx^2} + 2(1 + 2\sqrt{x})\frac{dy}{dx} - 3y = 0$$

giving your answer in the form $y = g(x)$. *(4 marks)*

END OF QUESTIONS

General Certificate of Education
Advanced Level Examination
January 2010

Mathematics MFP3

Unit Further Pure 3

Tuesday 19 January 2010 9.00 am to 10.30 am

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The **Examining Body** for this paper is AQA. The **Paper Reference** is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = x \ln(2x + y)$$

and $$y(3) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\,\mathrm{f}(x_r, y_r)$ and $k_2 = h\,\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places. *(5 marks)*

2 (a) Given that $y = \ln(4 + 3x)$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. *(3 marks)*

(b) Hence, by using Maclaurin's theorem, find the first three terms in the expansion, in ascending powers of x, of $\ln(4 + 3x)$. *(2 marks)*

(c) Write down the first three terms in the expansion, in ascending powers of x, of $\ln(4 - 3x)$. *(1 mark)*

(d) Show that, for small values of x,

$$\ln\left(\frac{4 + 3x}{4 - 3x}\right) \approx \frac{3}{2}x$$ *(2 marks)*

3 (a) A differential equation is given by

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x$$

Show that the substitution

$$u = \frac{dy}{dx}$$

transforms this differential equation into

$$\frac{du}{dx} + \frac{2}{x}u = 3$$

(2 marks)

(b) Find the general solution of

$$\frac{du}{dx} + \frac{2}{x}u = 3$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x$$

giving your answer in the form $y = g(x)$. *(2 marks)*

4 (a) Write down the expansion of $\sin 3x$ in ascending powers of x up to and including the term in x^3. *(1 mark)*

(b) Find

$$\lim_{x \to 0}\left[\frac{3x\cos 2x - \sin 3x}{5x^3}\right]$$

(4 marks)

Turn over ▶

5 It is given that y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\mathrm{e}^{-2x}$$

(a) Find the value of the constant p for which $y = px\mathrm{e}^{-2x}$ is a particular integral of the given differential equation. *(4 marks)*

(b) Solve the differential equation, expressing y in terms of x, given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. *(8 marks)*

6 (a) Explain why $\int_1^\infty \frac{\ln x^2}{x^3}\,\mathrm{d}x$ is an improper integral. *(1 mark)*

(b) (i) Show that the substitution $y = \frac{1}{x}$ transforms $\int \frac{\ln x^2}{x^3}\,\mathrm{d}x$ into $\int 2y \ln y\,\mathrm{d}y$. *(2 marks)*

(ii) Evaluate $\int_0^1 2y \ln y\,\mathrm{d}y$, showing the limiting process used. *(5 marks)*

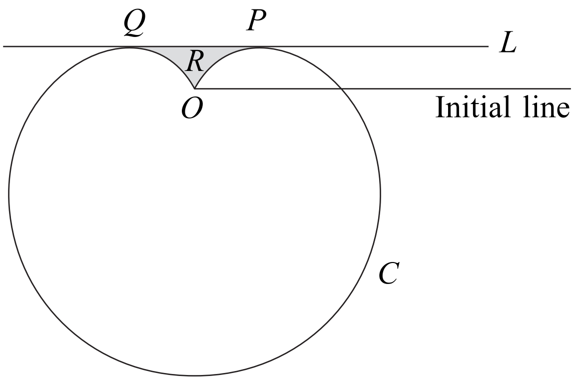
(iii) Hence write down the value of $\int_1^\infty \frac{\ln x^2}{x^3}\,\mathrm{d}x$. *(1 mark)*

7 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 8x^2 + 9\sin x$$

(8 marks)

8 The diagram shows a sketch of a curve C and a line L, which is parallel to the initial line and touches the curve at the points P and Q.



The polar equation of the curve C is

$$r = 4(1 - \sin\theta), \qquad 0 \leqslant \theta < 2\pi$$

and the polar equation of the line L is

$$r\sin\theta = 1$$

(a) Show that the polar coordinates of P are $\left(2, \frac{\pi}{6}\right)$ and find the polar coordinates of Q. *(5 marks)*

(b) Find the area of the shaded region R bounded by the line L and the curve C. Give your answer in the form $m\sqrt{3} + n\pi$, where m and n are integers. *(11 marks)*

END OF QUESTIONS

General Certificate of Education
Advanced Level Examination
June 2010

Mathematics MFP3

Unit Further Pure 3

Friday 11 June 2010 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = x + 3 + \sin y$$

and $$y(1) = 1$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

with your answer to part **(a)**, to obtain an approximation to $y(1.2)$, giving your answer to three decimal places. *(3 marks)*

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2 (a) Find the value of the constant k for which $k \sin 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + y = \sin 2x$$ *(3 marks)*

(b) Hence find the general solution of this differential equation. *(4 marks)*

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3 (a) Explain why $\int_1^\infty 4xe^{-4x}\,dx$ is an improper integral. *(1 mark)*

(b) Find $\int 4xe^{-4x}\,dx$. *(3 marks)*

(c) Hence evaluate $\int_1^\infty 4xe^{-4x}\,dx$, showing the limiting process used. *(3 marks)*

4 By using an integrating factor, find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x}y = (x^4 + 3)^{\frac{3}{2}}$$

given that $y = \frac{1}{5}$ when $x = 1$. *(9 marks)*

---

5 (a) Write down the expansion of $\cos 4x$ in ascending powers of x up to and including the term in x^4. Give your answer in its simplest form. *(2 marks)*

(b) (i) Given that $y = \ln(2 - \mathrm{e}^x)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

(You may leave your expression for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ unsimplified.) *(6 marks)*

(ii) Hence, by using Maclaurin's theorem, show that the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(2 - \mathrm{e}^x)$ are

$$-x - x^2 - x^3$$

(2 marks)

(c) Find

$$\lim_{x \to 0} \left[\frac{x \ln(2 - \mathrm{e}^x)}{1 - \cos 4x} \right]$$

(3 marks)

---

6 The polar equation of a curve C_1 is

$$r = 2(\cos\theta - \sin\theta), \quad 0 \leqslant \theta \leqslant 2\pi$$

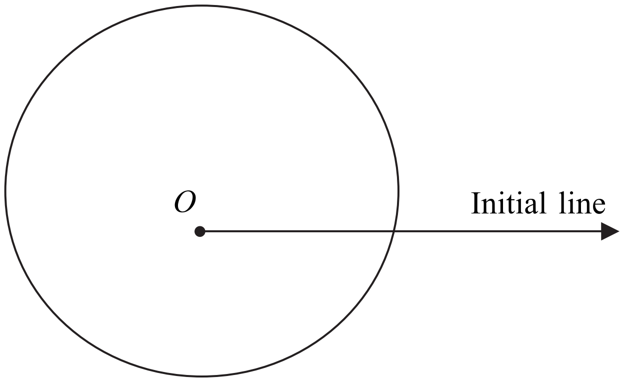
(a) (i) Find the cartesian equation of C_1. *(4 marks)*

(ii) Deduce that C_1 is a circle and find its radius and the cartesian coordinates of its centre. *(3 marks)*

Turn over ▶

(b) The diagram shows the curve C_2 with polar equation

$$r = 4 + \sin\theta, \quad 0 \leqslant \theta \leqslant 2\pi$$



(i) Find the area of the region that is bounded by C_2. *(6 marks)*

(ii) Prove that the curves C_1 and C_2 do not intersect. *(4 marks)*

(iii) Find the area of the region that is outside C_1 but inside C_2. *(2 marks)*

---

7 (a) Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$ and y is a function of x, show that:

(i) $\dfrac{dy}{dx} = 2t^{\frac{1}{2}}\dfrac{dy}{dt}$; *(2 marks)*

(ii) $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$. *(3 marks)*

(b) Hence show that the substitution $x = t^{\frac{1}{2}}$ transforms the differential equation

$$x\frac{d^2y}{dx^2} - (8x^2 + 1)\frac{dy}{dx} + 12x^3y = 12x^5$$

into

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t$$

(2 marks)

(c) Hence find the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - (8x^2 + 1)\frac{dy}{dx} + 12x^3y = 12x^5$$

giving your answer in the form $y = f(x)$. *(7 marks)*

END OF QUESTIONS

General Certificate of Education
Advanced Level Examination
January 2011

Mathematics MFP3

Unit Further Pure 3

Monday 24 January 2011 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = x + \sqrt{y}$$

and $$y(3) = 4$$

Use the improved Euler formula

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to three decimal places. *(5 marks)*

---

2 (a) Find the values of the constants p and q for which $p\sin x + q\cos x$ is a particular integral of the differential equation

$$\frac{dy}{dx} + 5y = 13\cos x$$ *(3 marks)*

(b) Hence find the general solution of this differential equation. *(3 marks)*

---

3 A curve C has polar equation $r(1 + \cos\theta) = 2$.

(a) Find the cartesian equation of C, giving your answer in the form $y^2 = f(x)$. *(5 marks)*

(b) The straight line with polar equation $4r = 3\sec\theta$ intersects the curve C at the points P and Q. Find the length of PQ. *(4 marks)*

---

4 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} - \frac{2}{x}y = 2x^3e^{2x}$$

given that $y = e^4$ when $x = 2$. Give your answer in the form $y = f(x)$. *(9 marks)*

---

5 (a) Write $\frac{4}{4x+1} - \frac{3}{3x+2}$ in the form $\frac{C}{(4x+1)(3x+2)}$, where C is a constant. *(1 mark)*

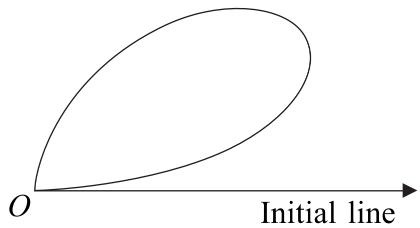
(b) Evaluate the improper integral

$$\int_1^{\infty} \frac{10}{(4x+1)(3x+2)}\,\mathrm{d}x$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant. *(6 marks)*

---

6 The diagram shows a sketch of a curve C.



The polar equation of the curve is

$$r = 2\sin 2\theta\sqrt{\cos\theta}\,, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

Show that the area of the region bounded by C is $\frac{16}{15}$. *(7 marks)*

---

7 (a) Write down the expansions in ascending powers of x up to and including the term in x^3 of:

(i) $\cos x + \sin x$; *(1 mark)*

(ii) $\ln(1+3x)$. *(1 mark)*

(b) It is given that $y = \mathrm{e}^{\tan x}$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (1+\tan x)^2\frac{\mathrm{d}y}{\mathrm{d}x}$. *(5 marks)*

(ii) Find the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ when $x = 0$. *(2 marks)*

Turn over ▶

(iii) Hence, by using Maclaurin's theorem, show that the first four terms in the expansion, in ascending powers of x, of $e^{\tan x}$ are

$$1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[\frac{e^{\tan x} - (\cos x + \sin x)}{x \ln(1 + 3x)} \right]$$ *(3 marks)*

---

8 (a) Given that $x = e^t$ and that y is a function of x, show that

$$x\frac{dy}{dx} = \frac{dy}{dt}$$ *(2 marks)*

(b) Hence show that the substitution $x = e^t$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

into

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$ *(5 marks)*

(c) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$ *(6 marks)*

(d) Hence solve the differential equation $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2\ln x$, given that $y = \dfrac{3}{2}$ and $\dfrac{dy}{dx} = \dfrac{1}{2}$ when $x = 1$. *(5 marks)*

General Certificate of Education
Advanced Level Examination
June 2011

Mathematics MFP3

Unit Further Pure 3

Thursday 16 June 2011 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = x + \ln(1 + y)$$

and $$y(2) = 1$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.2$, to obtain an approximation to $y(2.2)$, giving your answer to four decimal places. *(5 marks)*

---

2 (a) Find the values of the constants p and q for which $p + qx\mathrm{e}^{-2x}$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 4 - 9\mathrm{e}^{-2x}$$ *(5 marks)*

(b) Hence find the general solution of this differential equation. *(3 marks)*

(c) Hence express y in terms of x, given that $y = 4$ when $x = 0$ and that $\frac{\mathrm{d}y}{\mathrm{d}x} \to 0$ as $x \to \infty$. *(4 marks)*

---

3 (a) Find $\int x^2 \ln x \,\mathrm{d}x$. *(3 marks)*

(b) Explain why $\int_0^{\mathrm{e}} x^2 \ln x \,\mathrm{d}x$ is an improper integral. *(1 mark)*

(c) Evaluate $\int_0^{\mathrm{e}} x^2 \ln x \,\mathrm{d}x$, showing the limiting process used. *(3 marks)*

4 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\cot x)y = \sin 2x, \quad 0 < x < \frac{\pi}{2}$$

given that $y = \frac{1}{2}$ when $x = \frac{\pi}{6}$. *(10 marks)*

---

5 (a) Given that $y = \ln(1 + 2\tan x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(You may leave your expression for $\frac{d^2y}{dx^2}$ unsimplified.) *(4 marks)*

(b) Hence, using Maclaurin's theorem, find the first two non-zero terms in the expansion, in ascending powers of x, of $\ln(1 + 2\tan x)$. *(2 marks)*

(c) Find

$$\lim_{x \to 0} \left[\frac{\ln(1 + 2\tan x)}{\ln(1 - x)} \right]$$ *(4 marks)*

---

6 A differential equation is given by

$$(x^3 + 1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$$

(a) Show that the substitution

$$u = \frac{dy}{dx} - 2x$$

transforms this differential equation into

$$(x^3 + 1)\frac{du}{dx} = 3x^2 u$$ *(4 marks)*

(b) Hence find the general solution of the differential equation

$$(x^3 + 1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$$

giving your answer in the form $y = f(x)$. *(8 marks)*

7 The curve C_1 is defined by $r = 2\sin\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$.

The curve C_2 is defined by $r = \tan\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$.

(a) Find a cartesian equation of C_1. *(3 marks)*

(b) (i) Prove that the curves C_1 and C_2 meet at the pole O and at one other point, P, in the given domain. State the polar coordinates of P. *(4 marks)*

(ii) The point A is the point on the curve C_1 at which $\theta = \frac{\pi}{4}$.

The point B is the point on the curve C_2 at which $\theta = \frac{\pi}{4}$.

Determine which of the points A or B is further away from the pole O, justifying your answer. *(2 marks)*

(iii) Show that the area of the region bounded by the arc OP of C_1 and the arc OP of C_2 is $a\pi + b\sqrt{3}$, where a and b are rational numbers. *(10 marks)*

END OF QUESTIONS

General Certificate of Education
Advanced Level Examination
January 2012

Mathematics MFP3

Unit Further Pure 3

Monday 23 January 2012 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \frac{y - x}{y^2 + x}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\mathrm{f}(x_r, y_r)$$

with your answer to part **(a)**, to obtain an approximation to $y(1.2)$, giving your answer to three decimal places. *(3 marks)*

---

2 Find

$$\lim_{x \to 0} \left[\frac{\sqrt{4 + x} - 2}{x + x^2} \right]$$ *(3 marks)*

---

3 Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 26\mathrm{e}^x$$

given that $y = 5$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 11$ when $x = 0$. Give your answer in the form $y = \mathrm{f}(x)$. *(10 marks)*

4 (a) By using an integrating factor, find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = \ln x$$ *(7 marks)*

(b) Hence, given that $y \to 0$ as $x \to 0$, find the value of y when $x = 1$. *(3 marks)*

---

5 (a) Explain why $\int_{\frac{1}{2}}^{\infty} \frac{x(1-2x)}{x^2 + 3e^{4x}} \, dx$ is an improper integral. *(1 mark)*

(b) By using the substitution $u = x^2 e^{-4x} + 3$, find

$$\int \frac{x(1-2x)}{x^2 + 3e^{4x}} \, dx$$ *(3 marks)*

(c) Hence evaluate $\int_{\frac{1}{2}}^{\infty} \frac{x(1-2x)}{x^2 + 3e^{4x}} \, dx$, showing the limiting process used. *(4 marks)*

---

6 (a) Given that $y = \ln \cos 2x$, find $\frac{d^4y}{dx^4}$. *(6 marks)*

(b) Use Maclaurin's theorem to show that the first two non-zero terms in the expansion, in ascending powers of x, of $\ln \cos 2x$ are $-2x^2 - \frac{4}{3}x^4$. *(3 marks)*

(c) **Hence** find the first two non-zero terms in the expansion, in ascending powers of x, of $\ln \sec^2 2x$. *(2 marks)*

7 It is given that, for $x \neq 0$, y satisfies the differential equation

$$x\frac{d^2y}{dx^2} + 2(3x+1)\frac{dy}{dx} + 3y(3x+2) = 18x$$

(a) Show that the substitution $u = xy$ transforms this differential equation into

$$\frac{d^2u}{dx^2} + 6\frac{du}{dx} + 9u = 18x$$ *(4 marks)*

(b) Hence find the general solution of the differential equation

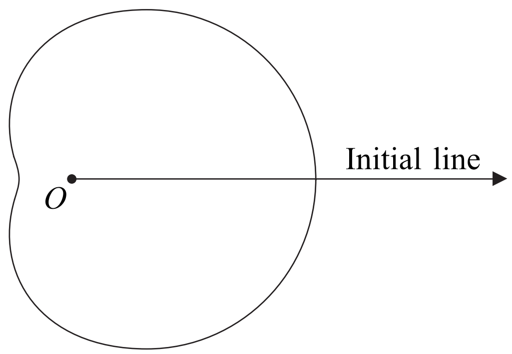
$$x\frac{d^2y}{dx^2} + 2(3x+1)\frac{dy}{dx} + 3y(3x+2) = 18x$$

giving your answer in the form $y = f(x)$. *(8 marks)*

---

8 The diagram shows a sketch of the curve C with polar equation

$$r = 3 + 2\cos\theta, \quad 0 \leqslant \theta \leqslant 2\pi$$



(a) Find the area of the region bounded by the curve C. *(6 marks)*

(b) A circle, whose cartesian equation is $(x-4)^2 + y^2 = 16$, intersects the curve C at the points A and B.

(i) Find, in surd form, the length of AB. *(6 marks)*

(ii) Find the perimeter of the segment AOB of the circle, where O is the pole. *(3 marks)*

General Certificate of Education
Advanced Level Examination
June 2012

Mathematics MFP3

Unit Further Pure 3

Thursday 14 June 2012 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
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- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \sqrt{(2x)} + \sqrt{y}$$

and $$y(2) = 9$$

Use the improved Euler formula

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.25$, to obtain an approximation to $y(2.25)$, giving your answer to two decimal places. *(5 marks)*

---

2 (a) Write down the expansion of $\sin 2x$ in ascending powers of x up to and including the term in x^5. *(1 mark)*

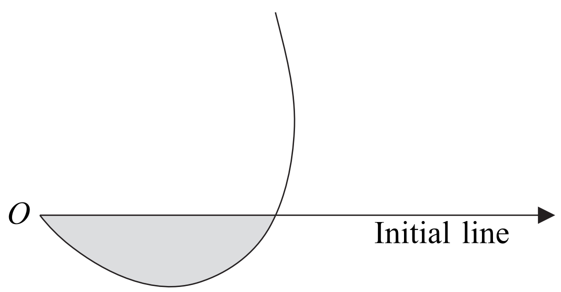
(b) Show that, for some value of k,

$$\lim_{x \to 0} \left[\frac{2x - \sin 2x}{x^2 \ln(1 + kx)} \right] = 16$$

and state this value of k. *(4 marks)*

---

3 The diagram shows a sketch of a curve C, the pole O and the initial line.



The polar equation of C is

$$r = 2\sqrt{1 + \tan\theta}, \qquad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$$

Show that the area of the shaded region, bounded by the curve C and the initial line, is $\frac{\pi}{2} - \ln 2$. *(4 marks)*

4 (a) By using an integrating factor, find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{4}{2x+1}y = 4(2x+1)^5$$

giving your answer in the form $y = f(x)$. *(7 marks)*

(b) The gradient of a curve at any point (x, y) on the curve is given by the differential equation

$$\frac{dy}{dx} = 4(2x+1)^5 - \frac{4}{2x+1}y$$

The point whose x-coordinate is zero is a stationary point of the curve. Using your answer to part **(a)**, find the equation of the curve. *(3 marks)*

---

5 (a) Find $\int x^2 e^{-x}\, dx$. *(4 marks)*

(b) Hence evaluate $\int_0^{\infty} x^2 e^{-x}\, dx$, showing the limiting process used. *(3 marks)*

---

6 It is given that $y = \ln(1 + \sin x)$.

(a) Find $\frac{dy}{dx}$. *(2 marks)*

(b) Show that $\frac{d^2y}{dx^2} = -e^{-y}$. *(3 marks)*

(c) Express $\frac{d^4y}{dx^4}$ in terms of $\frac{dy}{dx}$ and e^{-y}. *(3 marks)*

(d) Hence, by using Maclaurin's theorem, find the first four non-zero terms in the expansion, in ascending powers of x, of $\ln(1 + \sin x)$. *(3 marks)*

7 (a) Show that the substitution $x = e^t$ transforms the differential equation

$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 3 + 20\sin(\ln x)$$

into $$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 3 + 20\sin t$$ *(7 marks)*

(b) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 3 + 20\sin t$$ *(11 marks)*

(c) Write down the general solution of the differential equation

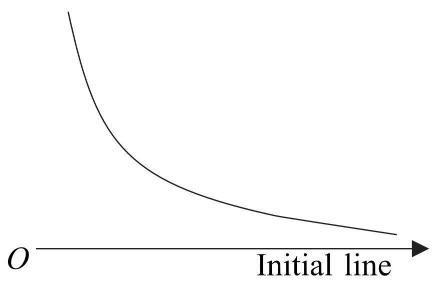
$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 3 + 20\sin(\ln x)$$ *(1 mark)*

---

8 (a) A curve has cartesian equation $xy = 8$. Show that the polar equation of the curve is $r^2 = 16\operatorname{cosec} 2\theta$. *(3 marks)*

(b) The diagram shows a sketch of the curve, C, whose polar equation is

$$r^2 = 16\operatorname{cosec} 2\theta, \quad 0 < \theta < \frac{\pi}{2}$$



(i) Find the polar coordinates of the point N which lies on the curve C and is closest to the pole O. *(2 marks)*

(ii) The circle whose polar equation is $r = 4\sqrt{2}$ intersects the curve C at the points P and Q. Find, in an exact form, the polar coordinates of P and Q. *(4 marks)*

(iii) The obtuse angle PNQ is α radians. Find the value of α, giving your answer to three significant figures. *(5 marks)*

General Certificate of Education
Advanced Level Examination
January 2013

Mathematics MFP3

Unit Further Pure 3

Friday 25 January 2013 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 It is given that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = \sqrt{2x + y}$

and $y(3) = 5$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.2$, to obtain an approximation to $y(3.2)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

with your answer to part **(a)**, to obtain an approximation to $y(3.4)$, giving your answer to three decimal places. *(3 marks)*

---

2 (a) Write down the expansion of e^{3x} in ascending powers of x up to and including the term in x^2. *(1 mark)*

(b) Hence, or otherwise, find the term in x^2 in the expansion, in ascending powers of x, of $e^{3x}(1 + 2x)^{-\frac{3}{2}}$. *(4 marks)*

---

3 It is given that the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

is $y = e^x(Ax + B)$. Hence find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6e^x$$ *(5 marks)*

4 (a) Explain why $\int_0^1 x^4 \ln x \, \mathrm{d}x$ is an improper integral. *(1 mark)*

(b) Evaluate $\int_0^1 x^4 \ln x \, \mathrm{d}x$, showing the limiting process used. *(6 marks)*

---

5 (a) Show that $\tan x$ is an integrating factor for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sec^2 x}{\tan x} y = \tan x$$

(2 marks)

(b) Hence solve this differential equation, given that $y = 3$ when $x = \frac{\pi}{4}$. *(6 marks)*

---

6 (a) It is given that $y = \ln(\mathrm{e}^{3x} \cos x)$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - \tan x$. *(3 marks)*

(ii) Find $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$. *(3 marks)*

(b) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\mathrm{e}^{3x} \cos x)$ are $3x - \frac{1}{2}x^2 - \frac{1}{12}x^4$. *(3 marks)*

(c) Write down the expansion of $\ln(1 + px)$, where p is a constant, in ascending powers of x up to and including the term in x^2. *(1 mark)*

(d) (i) Find the value of p for which $\lim_{x \to 0} \left[\frac{1}{x^2} \ln\left(\frac{\mathrm{e}^{3x} \cos x}{1 + px} \right) \right]$ exists.

(ii) Hence find the value of $\lim_{x \to 0} \left[\frac{1}{x^2} \ln\left(\frac{\mathrm{e}^{3x} \cos x}{1 + px} \right) \right]$ when p takes the value found in part **(d)(i)**. *(4 marks)*

7 (a) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = e^{2t}$$

giving your answer in the form $y = f(t)$. *(6 marks)*

(b) Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$ and y is a function of x, show that

$$\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$$ *(5 marks)*

(c) Hence show that the substitution $x = t^{\frac{1}{2}}$ transforms the differential equation

$$x\frac{d^2y}{dx^2} - (12x^2 + 1)\frac{dy}{dx} + 40x^3y = 4x^3e^{2x^2}$$

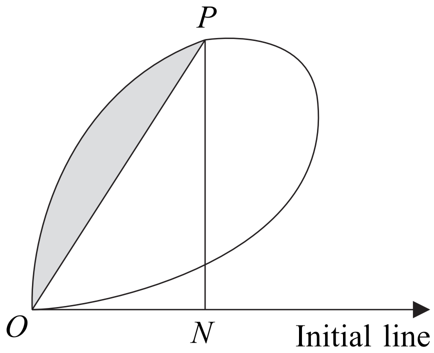
into

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = e^{2t}$$ *(2 marks)*

(d) Hence **write down** the general solution of the differential equation

$$x\frac{d^2y}{dx^2} - (12x^2 + 1)\frac{dy}{dx} + 40x^3y = 4x^3e^{2x^2}$$ *(1 mark)*

8 The diagram shows a sketch of a curve.



The polar equation of the curve is

$$r = \sin 2\theta \sqrt{\left(2 + \frac{1}{2}\cos\theta\right)}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The point P is the point of the curve at which $\theta = \frac{\pi}{3}$.

The perpendicular from P to the initial line meets the initial line at the point N.

(a) (i) Find the exact value of r when $\theta = \frac{\pi}{3}$. *(2 marks)*

(ii) Show that the polar equation of the line PN is $r = \frac{3\sqrt{3}}{8}\sec\theta$. *(2 marks)*

(iii) Find the area of triangle ONP in the form $\frac{k\sqrt{3}}{128}$, where k is an integer. *(2 marks)*

(b) (i) Using the substitution $u = \sin\theta$, or otherwise, find $\int \sin^n\theta\cos\theta\,\mathrm{d}\theta$, where $n \geqslant 2$. *(2 marks)*

(ii) Find the area of the shaded region bounded by the line OP and the arc OP of the curve. Give your answer in the form $a\pi + b\sqrt{3} + c$, where a, b and c are constants. *(8 marks)*

General Certificate of Education
Advanced Level Examination
June 2013

Mathematics MFP3

Unit Further Pure 3

Monday 10 June 2013 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
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- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 It is given that $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = (x - y)\sqrt{x + y}$$

and $$y(2) = 1$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.2$, to obtain an approximation to $y(2.2)$, giving your answer to three decimal places. *(5 marks)*

---

2 The Cartesian equation of a circle is $(x + 8)^2 + (y - 6)^2 = 100$.

Using the origin O as the pole and the positive x-axis as the initial line, find the polar equation of this circle, giving your answer in the form $r = p\sin\theta + q\cos\theta$. *(4 marks)*

---

3 (a) Find the values of the constants a, b and c for which $a + bx + cxe^{-3x}$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 3x - 8e^{-3x}$$ *(5 marks)*

(b) Hence find the general solution of this differential equation. *(3 marks)*

(c) Hence express y in terms of x, given that $y = 1$ when $x = 0$ and that $\frac{\mathrm{d}y}{\mathrm{d}x} \to -1$ as $x \to \infty$. *(4 marks)*

---

4 Evaluate the improper integral

$$\int_0^{\infty} \left(\frac{2x}{x^2 + 4} - \frac{4}{2x + 3} \right) \mathrm{d}x$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant. *(6 marks)*

5 (a) Differentiate $\ln(\ln x)$ with respect to x. *(1 mark)*

(b) (i) Show that $\ln x$ is an integrating factor for the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x \ln x} y = 9x^2, \quad x > 1$$ *(2 marks)*

(ii) Hence find the solution of this differential equation, given that $y = 4\mathrm{e}^3$ when $x = \mathrm{e}$. *(6 marks)*

---

6 It is given that $y = (4 + \sin x)^{\frac{1}{2}}$.

(a) Express $y\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\cos x$. *(2 marks)*

(b) Find the value of $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ when $x = 0$. *(5 marks)*

(c) Hence, by using Maclaurin's theorem, find the first four terms in the expansion, in ascending powers of x, of $(4 + \sin x)^{\frac{1}{2}}$. *(2 marks)*

---

7 A differential equation is given by

$$\sin^2 x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\sin x \cos x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\sin^4 x \cos x, \quad 0 < x < \pi$$

(a) Show that the substitution

$$y = u \sin x$$

where u is a function of x, transforms this differential equation into

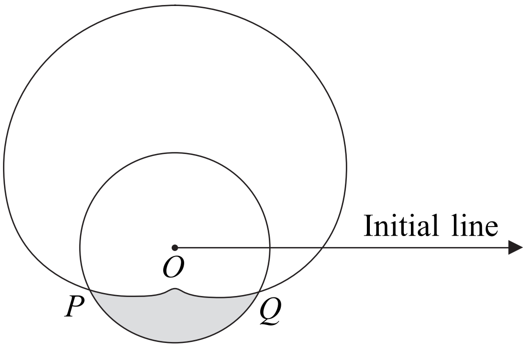
$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u = \sin 2x$$ *(5 marks)*

(b) Hence find the general solution of the differential equation

$$\sin^2 x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\sin x \cos x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\sin^4 x \cos x$$

giving your answer in the form $y = \mathrm{f}(x)$. *(6 marks)*

8 The diagram shows a sketch of a curve and a circle.



The polar equation of the curve is

$$r = 3 + 2\sin\theta, \quad 0 \leqslant \theta \leqslant 2\pi$$

The circle, whose polar equation is $r = 2$, intersects the curve at the points P and Q, as shown in the diagram.

(a) Find the polar coordinates of P and the polar coordinates of Q. *(3 marks)*

(b) A straight line, drawn from the point P through the pole O, intersects the curve again at the point A.

(i) Find the polar coordinates of A. *(2 marks)*

(ii) Find, in surd form, the length of AQ. *(3 marks)*

(iii) Hence, or otherwise, explain why the line AQ is a tangent to the circle $r = 2$. *(2 marks)*

(c) Find the area of the shaded region which lies inside the circle $r = 2$ but outside the curve $r = 3 + 2\sin\theta$. Give your answer in the form $\frac{1}{6}\left(m\sqrt{3} + n\pi\right)$, where m and n are integers. *(9 marks)*

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Surname										
Other Names										
Candidate Signature										

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TOTAL	

General Certificate of Education
Advanced Level Examination
June 2014

Mathematics MFP3

Unit Further Pure 3

Monday 19 May 2014 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
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Information
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- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

Answer **all** questions.

Answer each question in the space provided for that question.

1 It is given that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \frac{\ln(x + y)}{\ln y}$$

and $$y(6) = 3$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.4$, to obtain an approximation to $y(6.4)$, giving your answer to three decimal places.

[5 marks]

QUESTION PART REFERENCE

Answer space for question 1

2 (a) Find the values of the constants a, b and c for which $a + b\sin 2x + c\cos 2x$ is a particular integral of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 20 - 20\cos 2x$$

[4 marks]

(b) Hence find the solution of this differential equation, given that $y = 4$ when $x = 0$.

[4 marks]

QUESTION PART REFERENCE

Answer space for question 2

3 A curve has polar equation $r(4 - 3\cos\theta) = 4$. Find its Cartesian equation in the form $y^2 = \mathrm{f}(x)$.

[4 marks]

QUESTION PART REFERENCE

Answer space for question 3

4 Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{-x}$$

given that $y \to 0$ as $x \to \infty$ and that $\frac{dy}{dx} = -3$ when $x = 0$.

[10 marks]

QUESTION PART REFERENCE

Answer space for question 4

5 (a) Find $\int x \cos 8x \, \mathrm{d}x$.

[3 marks]

(b) Find $\lim_{x \to 0} \left[\frac{1}{x} \sin 2x\right]$.

[2 marks]

(c) Explain why $\int_0^{\frac{\pi}{4}} \left(2 \cot 2x - \frac{1}{x} + x \cos 8x\right) \mathrm{d}x$ is an improper integral.

[1 mark]

(d) Evaluate $\int_0^{\frac{\pi}{4}} \left(2 \cot 2x - \frac{1}{x} + x \cos 8x\right) \mathrm{d}x$, showing the limiting process used. Give your answer as a single term.

[4 marks]

QUESTION PART REFERENCE

Answer space for question 5

6 (a) By using an integrating factor, find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2x}{x^2+4}u = 3(x^2+4)$$

giving your answer in the form $u = \mathrm{f}(x)$.

[6 marks]

(b) Show that the substitution $u = x^2\frac{\mathrm{d}y}{\mathrm{d}x}$ transforms the differential equation

$$x^2(x^2+4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8x\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x^2+4)^2$$

into

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2x}{x^2+4}u = 3(x^2+4)$$

[4 marks]

(c) Hence, given that $x > 0$, find the general solution of the differential equation

$$x^2(x^2+4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8x\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x^2+4)^2$$

[2 marks]

QUESTION PART REFERENCE

Answer space for question 6

7 (a) It is given that $y = \ln(\cos x + \sin x)$.

(i) Show that $\dfrac{d^2y}{dx^2} = -\dfrac{2}{1 + \sin 2x}$.

[4 marks]

(ii) Find $\dfrac{d^3y}{dx^3}$.

[1 mark]

(b) (i) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\cos x + \sin x)$ are $x - x^2 + \frac{2}{3}x^3$.

[3 marks]

(ii) Write down the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\cos x - \sin x)$.

[1 mark]

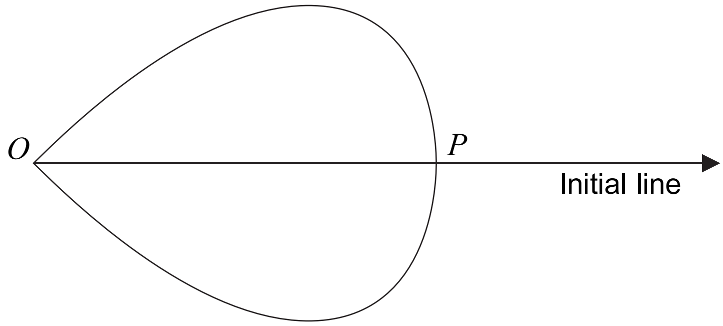
(c) **Hence** find the first three non-zero terms in the expansion, in ascending powers of x, of $\ln\left(\dfrac{\cos 2x}{e^{3x-1}}\right)$.

[4 marks]

QUESTION PART REFERENCE

Answer space for question 7

8 The diagram shows a sketch of a curve C, the pole O and the initial line. The curve C intersects the initial line at the point P.



The polar equation of C is $r = (1 - \tan^2 \theta) \sec \theta$, $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$.

(a) Show that the area of the region bounded by the curve C is $\frac{8}{15}$.

[5 marks]

(b) The curve whose polar equation is

$$r = \frac{1}{2}\sec^3 \theta, \quad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$$

intersects C at the points A and B.

(i) Find the polar coordinates of A and B.

[3 marks]

(ii) Given that $\text{angle } OAP = \text{angle } OBP = \alpha$, show that $\tan \alpha = k\sqrt{3}$, where k is an integer.

[4 marks]

(iii) Using your value of k from part **(b)(ii)**, state whether the point A lies inside or lies outside the circle whose diameter is OP. Give a reason for your answer.

[1 mark]

QUESTION PART REFERENCE	**Answer space for question 8**

Centre Number						Candidate Number				
Surname										
Other Names										
Candidate Signature										

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TOTAL	

General Certificate of Education
Advanced Level Examination
June 2015

Mathematics MFP3

Unit Further Pure 3

Wednesday 13 May 2015 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

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Answer **all** questions.

Answer each question in the space provided for that question.

1 It is given that $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \frac{x + y^2}{x}$$

and $$y(2) = 5$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

with $h = 0.05$, to obtain an approximation to $y(2.05)$.

[2 marks]

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\mathrm{f}(x_r, y_r)$$

with your answer to part **(a)**, to obtain an approximation to $y(2.1)$, giving your answer to three significant figures.

[3 marks]

QUESTION PART REFERENCE	**Answer space for question 1**

2 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\tan x)y = \tan^3 x \sec x$$

given that $y = 2$ when $x = \frac{\pi}{3}$.

[9 marks]

QUESTION PART REFERENCE

Answer space for question 2

3 (a) (i) Write down the expansion of $\ln(1+2x)$ in ascending powers of x up to and including the term in x^4.

[1 mark]

(ii) Hence, or otherwise, find the first two non-zero terms in the expansion of

$$\ln[(1+2x)(1-2x)]$$

in ascending powers of x and state the range of values of x for which the expansion is valid.

[3 marks]

(b) Find $\lim_{x \to 0} \left[\frac{3x - x\sqrt{9+x}}{\ln[(1+2x)(1-2x)]} \right]$.

[4 marks]

QUESTION PART REFERENCE

Answer space for question 3

4 (a) Explain why $\int_2^{\infty} (x-2)\mathrm{e}^{-2x}\,\mathrm{d}x$ is an improper integral.

[1 mark]

(b) Evaluate $\int_2^{\infty} (x-2)\mathrm{e}^{-2x}\,\mathrm{d}x$, showing the limiting process used.

[6 marks]

QUESTION PART REFERENCE

Answer space for question 4

5 (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 36\sin 3x$$

[7 marks]

(b) It is given that $y = f(x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 36\sin 3x$$

such that $f(0) = 0$ and $f'(0) = 0$.

(i) Show that $f''(0) = 0$.

[1 mark]

(ii) Find the first two non-zero terms in the expansion, in ascending powers of x, of $f(x)$.

[3 marks]

QUESTION PART REFERENCE

Answer space for question 5

6 A differential equation is given by

$$4\sqrt{x^5}\,\frac{d^2y}{dx^2} + (2\sqrt{x})y = \sqrt{x}(\ln x)^2 + 5, \quad x > 0$$

(a) Show that the substitution $x = e^{2t}$ transforms this differential equation into

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 4t^2 + 5e^{-t}$$

[7 marks]

(b) Hence find the general solution of the differential equation

$$4\sqrt{x^5}\,\frac{d^2y}{dx^2} + (2\sqrt{x})y = \sqrt{x}(\ln x)^2 + 5, \quad x > 0$$

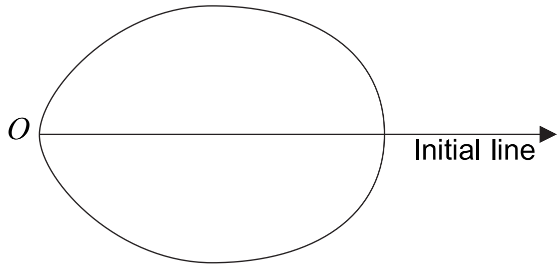
[10 marks]

QUESTION PART REFERENCE

Answer space for question 6

Do not write outside the box

7 The diagram shows the sketch of a curve C_1.



The polar equation of the curve C_1 is

$$r = 1 + \cos 2\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$

(a) Find the area of the region bounded by the curve C_1.

[5 marks]

(b) The curve C_2 whose polar equation is

$$r = 1 + \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$

intersects the curve C_1 at the pole O and at the point A. The straight line drawn through A parallel to the initial line intersects C_1 again at the point B.

(i) Find the polar coordinates of A.

[4 marks]

(ii) Show that the length of OB is $\frac{1}{4}\left(\sqrt{13}+1\right)$.

[6 marks]

(iii) Find the length of AB, giving your answer to three significant figures.

[3 marks]

QUESTION PART REFERENCE

Answer space for question 7